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**Abstract:** We give a new description of the data needed to specify a morphism from a scheme to a toric Deligne-Mumford stack. The description is given in terms of a collection of line bundles and sections which satisfy certain conditions. As applications, we characterize any toric Deligne-Mumford stack as a product of roots of line bundles over the rigidified stack, describe the torus action, describe morphisms between toric Deligne-Mumford stacks with complete coarse moduli spaces in terms of homogeneous polynomials, and compare two different definitions of toric stacks.

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# A NOTE ON TORIC DELIGNE-MUMFORD STACKS

FABIO PERRONI

**ABSTRACT.** We give a new description of the data needed to specify a morphism from a scheme to a toric DM-stack. The description is given in terms of a collection of line bundles and sections which satisfy certain conditions. As an application of this result we get some geometric properties of toric DM-stacks (including the torus action), and we describe morphisms between toric DM-stacks with complete coarse moduli spaces in terms of homogeneous polynomials.

## 1. INTRODUCTION

A map from a scheme  $Y$  to the projective space  $\mathbb{P}^d$  is determined by a line bundle  $L$  on  $Y$  together with  $d + 1$  sections which do not vanish simultaneously. More generally, when  $X$  is a smooth toric variety a map  $Y \rightarrow X$  is determined by a collection of line bundles and sections on  $Y$  which satisfy certain compatibility and nondegeneracy conditions [6]. An analogous result holds in the case of a simplicial toric variety  $X$ : consider the natural orbifold structure  $\mathcal{X}$  on  $X$ , then maps  $Y \rightarrow \mathcal{X}$  are determined by collections of line bundles and sections on  $Y$  precisely as in the case of  $X$  smooth [7]. More recently, toric Deligne-Mumford stacks have been defined in a slightly different way [4]. This definition takes into account also toric stacks with the property that the automorphism group of the generic point is not trivial.

The goal of the present paper is to generalize the results in [6] and [7] in order to describe morphisms  $Y \rightarrow \mathcal{X}$ , where  $\mathcal{X}$  is a toric DM-stack in the sense of [4] (Thm. 2.5).

As an application of this result we deduce some geometric properties of toric DM-stacks. We show that, given  $\mathcal{X}$ , the rigidification with respect to the generic automorphism group  $\mathcal{X} \rightarrow \mathcal{X}_{\text{rig}}$  is isomorphic to the fibered product of roots of certain line bundles over  $\mathcal{X}_{\text{rig}}$  (Pro. 3.2). This result is used to obtain a classification of toric DM-stacks in terms of the combinatorial data  $\Delta$  (Thm. 3.7). In Sec.4 we describe the torus action. Finally, we show how homogeneous polynomials can be used to describe all maps  $\mathcal{Y} \rightarrow \mathcal{X}$  between the toric DM-stacks  $\mathcal{Y}$  and  $\mathcal{X}$  whose coarse moduli spaces are complete varieties (Thm. 5.1).

The following notation is fixed through all the paper. We denote by  $\Delta$  the following set of data:

- a free abelian group  $N$  of rank  $d$ ;
- a rational simplicial fan  $\Delta$  in  $N_{\mathbb{Q}}$ , where  $N_{\mathbb{Q}} := N \otimes_{\mathbb{Z}} \mathbb{Q}$ ;
- an element  $a_{\rho} \in \rho \cap N$  for any  $\rho \in \Delta(1)$ , where  $\Delta(1)$  is the set of 1-dimensional cones of  $\Delta$ ;

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- a sequence of  $R$  positive non zero integers  $r_1, \dots, r_R$ ;
- and an integer  $b_{i\rho} \in \mathbb{Z}$ , for any  $i \in \{1, \dots, R\}$  and  $\rho \in \Delta(1)$ .

We denote with  $M$  the dual lattice of  $N$ ,  $M := \text{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$ , and  $\Delta_{\max}$  is the set of maximal cones in  $\Delta$  with respect to the partial relation  $\subseteq$ . We will use  $\otimes_{\rho}$  to denote  $\otimes_{\rho \in \Delta(1)}$ ,  $\otimes_i$  to denote  $\otimes_{i=1}^R$ , and similarly for  $\sum_{\rho}$  and  $\sum_i$ .

We will work over the field of complex numbers  $\mathbb{C}$ , hence any scheme will be over  $\mathbb{C}$  and any morphism of schemes will be a morphism of  $\mathbb{C}$ -schemes. This category is denoted by  $(\text{Sch})$ . For any scheme  $Y$ , we denote by  $\underline{\mathbb{C}}$  the trivial line bundle over it.

## 2. $\Delta$ -COLLECTIONS.

$\Delta$ -collections were introduced in [6] as the data needed to specify a morphism from a scheme to a smooth toric variety (see Example 2.3). The generalization to the case of a simplicial fan  $\Delta$  has been studied in [7]: when  $R = 0$  the content of Thm. 2.5 below coincides with that of Thm. 16 in [7]. In this Section we investigate the case of morphisms to a toric DM-stack (in the sense of [4]). We define  $\Delta$ -collections and morphisms between them as collections of line bundles and sections over a given scheme which satisfy certain compatibility and nondegeneracy conditions. The main result is Thm. 2.5 where we prove that the category of  $\Delta$ -collections  $\mathcal{C}_{\Delta}$  is a smooth DM-stack isomorphic to the toric DM-stack  $\mathcal{X}_{\Sigma}$ , where  $\Sigma$  is a stacky fan determined by  $\Delta$ .

**Definition 2.1.** Let  $Y$  be a scheme. A  $\Delta$ -collection on  $Y$  is the data of a line bundle  $L_{\rho}$  and a section  $u_{\rho} \in H^0(Y, L_{\rho})$  for any  $\rho \in \Delta(1)$ , isomorphisms

$$c_m : \otimes_{\rho} L_{\rho}^{\otimes \langle m, a_{\rho} \rangle} \rightarrow \underline{\mathbb{C}} \quad \text{for any } m \in M,$$

a line bundle  $M_i$  and an isomorphism

$$d_i : \otimes_{\rho} L_{\rho}^{\otimes b_{i\rho}} \otimes M_i^{\otimes r_i} \rightarrow \underline{\mathbb{C}} \quad \text{for any } i \in \{1, \dots, R\},$$

such that the following conditions are satisfied:

- (1)  $c_m \otimes c_{m'} = c_{m+m'}$  for any  $m, m' \in M$ ;
- (2) for any  $y \in Y$ , there exists a cone  $\sigma \in \Delta_{\max}$  such that  $u_{\rho}(y) \neq 0$  for all  $\rho \notin \sigma$ .

A  $\Delta$ -collection on  $Y$  is written  $(L_{\rho}, u_{\rho}, c_m, M_i, d_i)/Y$ .

**Definition 2.2.** Let  $(L'_{\rho}, u'_{\rho}, c'_m, M'_i, d'_i)/Y'$  and  $(L_{\rho}, u_{\rho}, c_m, M_i, d_i)/Y$  be two  $\Delta$ -collections. A **morphism** from  $(L'_{\rho}, u'_{\rho}, c'_m, M'_i, d'_i)/Y'$  to  $(L_{\rho}, u_{\rho}, c_m, M_i, d_i)/Y$  is given by a morphism  $f : Y' \rightarrow Y$  of schemes, morphisms  $\gamma_{\rho} : L'_{\rho} \rightarrow L_{\rho}$  and  $\delta_i : M'_i \rightarrow M_i$  of line bundles for any  $\rho \in \Delta(1)$  and  $i \in \{1, \dots, R\}$ , such that the following conditions are satisfied:

- the  $\gamma_{\rho}$ 's induce isomorphisms  $L'_{\rho} \rightarrow f^* L_{\rho}$ , the same holds for the  $\delta_i$ 's;
- $\gamma_{\rho} \circ u'_{\rho} = u_{\rho} \circ f$  for all  $\rho \in \Delta(1)$ ;
- for any  $m \in M$  the following diagram commutes

$$\begin{array}{ccc} \otimes_{\rho} L'_{\rho}^{\otimes \langle m, a_{\rho} \rangle} & \xrightarrow{c'_m} & \underline{\mathbb{C}} \\ \otimes_{\rho} \gamma_{\rho}^{\otimes \langle m, a_{\rho} \rangle} \downarrow & & \downarrow f \times \text{id}_{\mathbb{C}} \\ \otimes_{\rho} L_{\rho}^{\otimes \langle m, a_{\rho} \rangle} & \xrightarrow{c_m} & \underline{\mathbb{C}} \end{array}$$

- for any  $i \in \{1, \dots, R\}$  the following diagram commutes

$$\begin{array}{ccc} \otimes_{\rho} L_{\rho}'^{\otimes b_{i\rho}} \otimes M_i'^{\otimes r_i} & \xrightarrow{d_i'} & \underline{\mathbb{C}} \\ \otimes_{\rho} \gamma_{\rho}^{\otimes b_{i\rho}} \otimes \delta_i^{\otimes r_i} \downarrow & & \downarrow f \times \text{id}_{\mathbb{C}} \\ \otimes_{\rho} L_{\rho}^{\otimes b_{i\rho}} \otimes M_i^{\otimes r_i} & \xrightarrow{d_i} & \underline{\mathbb{C}}. \end{array}$$

A morphism from  $(L'_{\rho}, u'_{\rho}, c'_m, M'_i, d'_i)/Y'$  to  $(L_{\rho}, u_{\rho}, c_m, M_i, d_i)/Y$  is denoted by  $(f, \gamma_{\rho}, \delta_i)$ .

Let us consider some examples.

**Example 2.3.** Let  $\Delta$  be a fan and  $N$  be a lattice which determine a smooth toric variety  $X$ . Set  $a_{\rho} := n_{\rho}$  be the minimal lattice points of the rays, and  $R = 0$ . In this case  $\Delta$  is determined by  $\Delta$  and  $N$ , so we talk about  $\Delta$ -collections. On  $X$  there is a canonical  $\Delta$ -collection defined as follows. Since  $X$  is smooth, for each  $\rho$ , the  $T$ -invariant divisor  $D_{\rho}$  gives a line bundle  $\mathcal{O}_X(D_{\rho})$ , moreover there is a natural global section  $\iota_{\rho} \in H^0(X, \mathcal{O}_X(D_{\rho}))$ . For any  $m \in M$ , the character  $\chi^m$  is a rational function on  $X$  such that  $\text{div}(\chi^m) = \sum_{\rho} \langle m, n_{\rho} \rangle D_{\rho}$ , hence we get an isomorphism

$$c_{\chi^m} : \otimes_{\rho} \mathcal{O}_X(D_{\rho})^{\otimes \langle m, n_{\rho} \rangle} \rightarrow \underline{\mathbb{C}}.$$

Then,  $(\mathcal{O}_X(D_{\rho}), \iota_{\rho}, c_{\chi^m})$  is a  $\Delta$ -collection on  $X$  (Lemma 1.1 in [6]). This  $\Delta$ -collection is called *universal* because of the following result. Let

$$C_{\Delta} : (\text{Sch}) \rightarrow (\text{Sets})$$

be the contravariant functor that associates to any scheme  $Y$  the set of equivalence classes of  $\Delta$ -collections on  $Y$ . Then  $X$  is the fine moduli space for  $C_{\Delta}$ , and  $(\mathcal{O}_X(D_{\rho}), \iota_{\rho}, c_{\chi^m})$  is the universal family ([6]).

**Example 2.4.**  $N := \mathbb{Z}$ ,  $\Delta := \{\{0\}, \rho := \mathbb{Q}_{\geq 0}\}$ ,  $a_{\rho} := a$ ,  $R := 0$ . A  $\Delta$ -collection is given by a line bundle  $L$  on  $Y$ , with a section  $u \in H^0(Y, L)$ , and an isomorphism  $c : L^{\otimes a} \rightarrow \underline{\mathbb{C}}$ . Then the category of  $\Delta$ -collections is equivalent to the stack  $[\mathbb{A}_{\mathbb{C}}^1/\mu_a]$ , where  $\mu_a$  is the group of  $a$ -th roots of 1 acting on the right on  $\mathbb{A}_{\mathbb{C}}^1$  by  $(x, \zeta) \mapsto x \cdot \zeta$ .

The main result of the present paper (Thm. 2.5) states that, in general, there is a correspondence between combinatorial data  $\Delta$  and toric DM-stacks. Let  $\mathcal{C}_{\Delta}$  be the category whose objects are  $\Delta$ -collections and morphisms are morphisms between  $\Delta$ -collections. The functor

$$p : \mathcal{C}_{\Delta} \rightarrow (\text{Sch})$$

which sends the  $\Delta$ -collection  $(L_{\rho}, u_{\rho}, c_m, M_i, d_i)/Y$  to  $Y$  and the morphism  $(f, \gamma_{\rho}, \delta_i)$  to  $f$  makes  $\mathcal{C}_{\Delta}$  a category fibered in groupoids (a CFGs) over  $(\text{Sch})$ .

**Theorem 2.5.**  $\mathcal{C}_{\Delta}$  is a smooth Deligne-Mumford stack whose coarse moduli space is the toric variety associated to the fan  $\Delta$  and the lattice  $N$ .

If the 1-dimensional cones  $\rho$  of  $\Delta$  span  $N_{\mathbb{Q}}$ , set

$$\Sigma := (N \oplus_{i=1}^R \mathbb{Z}/r_i, \Delta, \{(a_{\rho}, [b_{1\rho}]_{r_1}, \dots, [b_{R\rho}]_{r_R})\}_{\rho}),$$

where  $[b_{i\rho}]_{r_i}$  denotes the class of  $b_{i\rho}$  in  $\mathbb{Z}/r_i$ . Then  $\mathcal{C}_{\Delta}$  is isomorphic to the toric Deligne-Mumford stack  $\mathcal{X}_{\Sigma}$  associated to the stacky fan  $\Sigma$  as defined in [4].

We notice that one can define  $\Delta$ -collections over a stack formally in the same way as for schemes. In particular, on  $\mathcal{C}_\Delta$  there is a canonical  $\Delta$ -collection which is defined in the obvious way:

$$(\mathfrak{L}_\rho, \mathfrak{u}_\rho, \mathfrak{c}_m, \mathfrak{M}_i, \mathfrak{d}_i)/\mathcal{C}_\Delta.$$

As a result the theory of descent gives the following

**Corollary 2.6.** *Let  $\mathcal{Y}$  be a DM-stack. Then the category of morphisms  $\mathcal{Y} \rightarrow \mathcal{C}_\Delta$  is equivalent to the category of  $\Delta$ -collections over  $\mathcal{Y}$ .*

We collect below some general results we need in order to prove the above Theorem.

**Notation 2.7.** Following the notations used in [3], for any quasi-affine group scheme  $G$ , we denote by

$$p_G : BG \rightarrow (\text{Sch})$$

the structure morphism of the stack  $BG$ , and by

$$\pi_G : \text{Spec}\mathbb{C} \rightarrow BG$$

the covering. We denote by

$$b_G : \text{pre}BG \rightarrow BG$$

the stackification morphism from the pre-stack  $\text{pre}BG$  to  $BG$ . For any morphism  $\varphi : G \rightarrow H$  of group schemes, we denote by

$$\text{pre}B\varphi : \text{pre}BG \rightarrow \text{pre}BH$$

the induced morphism of pre-stacks.

We quote a remark from [3].

**Remark 2.8.** Let  $\varphi : G \rightarrow H$  be a morphism of quasi-affine group schemes. Let  $P \rightarrow Y$  be a principal  $G$ -bundle. Consider the right  $G$ -action on  $P \times H$ :

$$(2.9) \quad (x, h) \cdot g = (x \cdot g, \varphi(g^{-1}) \cdot h).$$

The theory of descent guarantees that there exists a quotient scheme for the previous action,  $(P \times H)/G$ , together with a morphism  $(P \times H)/G \rightarrow Y$  inducing a principal  $H$ -bundle structure. The scheme  $(P \times H)/G$  will be denoted either as  $P \times_\varphi H$  or by  $P \times_G H$ . Notice that  $P \times_\varphi H$  is not unique and due to this ambiguity the correspondence  $P \mapsto P \times_\varphi H$  defines a functor  $B\varphi : BG \rightarrow BH$  up to unique canonical 2-isomorphism. In the following this ambiguity will be understood.

An analogous notation will be used to denote the quotients  $P \times_G X$  in the case where  $X$  is a quasi-affine scheme with a right  $G$ -action.

We now recall two well known results in order to be self-contained as much as possible. For complex algebraic varieties the reference is [10]. For a more general context we refer to [8], Ch. III, Prop. 3.2.1.

**Lemma 2.10.** *Let*

$$1 \rightarrow G \xrightarrow{\varphi} H \xrightarrow{\psi} K \rightarrow 1$$

*be a short exact sequence of quasi-affine group schemes. Then*

$$(2.11) \quad (B\varphi, p_G) : BG \rightarrow BH \times_{B\psi \times \pi_K} \text{Spec}\mathbb{C}$$

*is an isomorphism of stacks.*

**Lemma 2.12.** *Let  $G$  be a quasi-affine group scheme,  $X$  a quasi-affine scheme with a right  $G$ -action. For any principal  $G$ -bundle  $\pi : P \rightarrow Y$ , there is a bijection*

$$(2.13) \quad \{\text{sections of } P \times_G X \rightarrow Y\} \leftrightarrow \{G\text{-equivariant morphisms } t : P \rightarrow X\}.$$

*which is functorial with respect to base change.*

**Proof of Thm. 2.5.** We first prove the case in which the set  $\{\rho \in \Delta(1)\}$  spans  $N_{\mathbb{Q}}$ . Identify the lattice  $N$  with  $\mathbb{Z}^d$  and enumerate the 1-dimensional cones of  $\Delta$  as  $\rho_1, \dots, \rho_n$ . Then the  $a_\rho \in N$  correspond to  $(a_{1k}, \dots, a_{dk}) \in \mathbb{Z}^d$ ,  $k \in \{1, \dots, n\}$ . Let us define the matrices

$$B := \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \dots & \vdots \\ a_{d1} & \dots & a_{dn} \\ b_{11} & \dots & b_{1n} \\ \vdots & \dots & \vdots \\ b_{R1} & \dots & b_{Rn} \end{pmatrix}; \quad Q := \begin{pmatrix} 0 & \dots & 0 \\ \vdots & \dots & \vdots \\ 0 & \dots & 0 \\ r_1 & \dots & 0 \\ \vdots & \dots & \vdots \\ 0 & \dots & r_R \end{pmatrix}$$

$B \in \text{Mat}((d+R) \times n, \mathbb{Z})$ ,  $Q \in \text{Mat}((d+R) \times R, \mathbb{Z})$ . We consider the exact sequence

$$0 \rightarrow (\mathbb{Z}^{d+R})^* \xrightarrow{[BQ]^*} (\mathbb{Z}^{n+R})^* \rightarrow \text{coker}([BQ]^*) \rightarrow 0,$$

and we apply the functor  $\text{Hom}_{\mathbb{Z}}(-, \mathbb{C}^*)$ , we get an exact sequence of affine group schemes:

$$(2.14) \quad 1 \rightarrow G \xrightarrow{\varphi} (\mathbb{C}^*)^n \times (\mathbb{C}^*)^R \xrightarrow{\psi} (\mathbb{C}^*)^d \times (\mathbb{C}^*)^R \rightarrow 1$$

where

$$(2.15) \quad \begin{aligned} & \psi(\lambda_1, \dots, \lambda_n, \mu_1, \dots, \mu_R) \\ &= (\lambda_1^{a_{11}} \dots \lambda_n^{a_{1n}}, \dots, \lambda_1^{a_{d1}} \dots \lambda_n^{a_{dn}}, \mu_1^{r_1} \cdot \lambda_1^{b_{11}} \dots \lambda_n^{b_{1n}}, \dots, \mu_R^{r_R} \cdot \lambda_1^{b_{R1}} \dots \lambda_n^{b_{Rn}}). \end{aligned}$$

The matrix  $Q$  defines a morphism  $\mathbb{Z}^R \rightarrow \mathbb{Z}^{n+R}$  which is a projective resolution of  $N \oplus_i \mathbb{Z}/r_i$ , and  $B$  defines a lifting  $\mathbb{Z}^n \rightarrow \mathbb{Z}^{d+R}$  of the morphism  $\mathbb{Z}^n \rightarrow N \oplus_i \mathbb{Z}/r_i$ ,  $e_i \mapsto (a_{\rho_i}, [b_{1\rho_i}]_{r_1}, \dots, [b_{R\rho_i}]_{r_R})$ , where  $e_i$  is the  $i$ -th element of the standard basis of  $\mathbb{Z}^n$ . Following [4] we associate to this data a toric DM-stack  $\mathcal{X}_{\Sigma} := [Z/G]$ .

We now define a functor of CFGs over  $(\text{Sch})$ ,

$$(2.16) \quad F : \mathcal{X}_{\Sigma} \rightarrow \mathcal{C}_{\Delta}.$$

Consider an object of  $\mathcal{X}_{\Sigma}(Y)$ ,

$$\begin{array}{ccc} P & \xrightarrow{t} & Z \\ \pi \downarrow & & \\ Y & & \end{array}$$

Set

$$L_k := B\varphi(P) \times_{(\mathbb{C}^*)^n \times (\mathbb{C}^*)^R} \mathbb{C},$$

the associate line bundle with respect to the action

$$\begin{aligned} \mathbb{C} \times ((\mathbb{C}^*)^n \times (\mathbb{C}^*)^R) & \rightarrow \mathbb{C} \\ z \cdot (\lambda_1, \dots, \lambda_n, \mu_1, \dots, \mu_R) & \mapsto z \cdot \lambda_k, \end{aligned}$$

where  $\varphi$  is defined in (2.14), and  $k \in \{1, \dots, n\}$ . In the same way, for any  $i \in \{1, \dots, R\}$ , set

$$M_i := B\varphi(P) \times_{(\mathbb{C}^*)^n \times (\mathbb{C}^*)^R} \mathbb{C},$$

be the line bundle associated to the action

$$\begin{aligned} \mathbb{C} \times ((\mathbb{C}^*)^n \times (\mathbb{C}^*)^R) &\rightarrow \mathbb{C} \\ z \cdot (\lambda_1, \dots, \lambda_n, \mu_1, \dots, \mu_R) &\mapsto z \cdot \mu_i. \end{aligned}$$

We now define the sections of  $L_1, \dots, L_n$ . Recall that  $G$  acts on  $\mathbb{C}^n$  by means of the component on  $(\mathbb{C}^*)^n$  of  $\varphi$ . Then the composition of  $t : P \rightarrow Z$  with the inclusion  $Z \rightarrow \mathbb{C}^n$  gives an equivariant morphism  $\tilde{t} : P \rightarrow \mathbb{C}^n$ . By Lemma 2.12 we get a section  $u$  of the vector bundle

$$(2.17) \quad P \times_G \mathbb{C}^n = L_1 \oplus \dots \oplus L_n,$$

then  $u_k$  is the  $k$ -th component of  $u$  with respect to the decomposition (2.17). Condition (2) of Def. 2.1 follows from the fact that  $t(P) \subset Z$  and from the definition of  $Z$  [4].

We next define the isomorphisms  $c_m$  required in the definition of  $\Delta$ -collection. By applying the functor  $(B\varphi, p_G)$  defined in Lemma 2.10, we get

$$(2.18) \quad (B\varphi, p_G)(P) = (B\varphi(P), \alpha, Y \times ((\mathbb{C}^*)^d \times (\mathbb{C}^*)^R))$$

where  $\alpha : B\psi(B\varphi(P)) \rightarrow Y \times ((\mathbb{C}^*)^d \times (\mathbb{C}^*)^R)$  is an isomorphism of principal  $((\mathbb{C}^*)^d \times (\mathbb{C}^*)^R)$ -bundles. Consider now the diagonal action of  $((\mathbb{C}^*)^d \times (\mathbb{C}^*)^R)$  on  $\mathbb{C}^{d+R}$  with weights 1. Then from (2.15) it follows that the associated vector bundle

$$B\psi(B\varphi(P)) \times_{(\mathbb{C}^*)^d \times (\mathbb{C}^*)^R} \mathbb{C}^{d+R}$$

is canonically isomorphic to

$$(2.19) \quad \bigoplus_{l=1}^d \left( \bigotimes_{k=1}^n L_k^{\otimes a_{kl}} \right) \oplus_i^r \left( M^{\otimes r_i} \bigotimes_{k=1}^n L^{\otimes b_{ik}} \right).$$

The isomorphism  $\alpha$  in (2.18) gives an isomorphism between (2.19) and the trivial vector bundle, then we get isomorphisms  $c_{e_1^*}, \dots, c_{e_d^*}, d_1, \dots, d_R$ , where  $e_1^*, \dots, e_d^*$  is the dual basis of the standard basis  $e_1, \dots, e_d$  of  $\mathbb{Z}^d$ . The  $c_m$ 's are now uniquely determined by condition (1) of Def. 2.1.

We have defined  $F$  on objects. The definition on morphisms and the verification of the fact that it is a functor is straightforward, using the theory of descent for quasi-affine morphisms. Moreover,  $F$  is an equivalence of categories. This follows from Lemmas 2.10 and 2.12, and from the equivalence between the category of principal  $\mathbb{C}^*$ -bundles and the one of line bundles, because  $\mathbb{C}^*$  is a special group [10] (see also Exercise 2.1 [3]). This, together with Prop. 3.2 and Prop. 3.7 of [4] completes the proof under the assumptions that the set  $\{\rho \in \Delta(1)\}$  spans  $N_{\mathbb{Q}}$ .

We next consider the case where  $\{\rho \in \Delta(1)\}$  does not span  $N_{\mathbb{Q}}$ . We follow the ideas used in the proof of Thm. 1.1 in [6]. Set

$$N' := \text{Span}(\{\rho \in \Delta(1)\}) \cap N.$$

The fan  $\Delta$  can be regarded as a fan in  $N'_{\mathbb{Q}}$ , then set

$$\Delta' = \{N', \Delta, \{a_\rho\}, r_1, \dots, r_R, \{b_{i\rho}\}\}.$$

From the first part of the proof we have that  $\mathcal{C}_{\Delta'}$  is a smooth DM-stack.

$N/N'$  is torsion free, so we can find a subgroup  $N''$  of  $N$  such that  $N = N' \oplus N''$ . The projection  $N \rightarrow N'$  determines an inclusion  $\iota : M' \rightarrow M$  such that  $M = \iota(M') \oplus N'^{\perp}$ .

Let now  $(L_\rho, u_\rho, c_m, M_i, d_i)/Y$  be a  $\Delta$ -collection. For any  $m \in N'^\perp$ , we have  $\langle m, a_\rho \rangle = 0$  for all  $\rho$ . Thus  $c_m$  can be identified with an element in  $H^0(Y, \mathcal{O}_Y^*)$ . Under this identification, the application  $N'^\perp \rightarrow H^0(Y, \mathcal{O}_Y^*)$ ,  $m \mapsto c_m$ , is a group homomorphism, thus induces a morphism of schemes  $Y \rightarrow \text{Spec}(\mathbb{C}[N'^\perp])$ . In this way we get a functor

$$(2.20) \quad \mathcal{C}_\Delta \rightarrow \text{Spec}(\mathbb{C}[N'^\perp]).$$

On the other hand there is an obvious functor

$$(2.21) \quad \mathcal{C}_\Delta \rightarrow \mathcal{C}_{\Delta'}$$

which associates  $(L_\rho, u_\rho, c_m, M_i, d_i)/Y$  to  $(L_\rho, u_\rho, \{c_m : m \in \iota(M')\}, M_i, d_i)/Y$ . Then the functor  $\mathcal{C}_\Delta \rightarrow \mathcal{C}_{\Delta'} \times \text{Spec}(\mathbb{C}[N'^\perp])$  whose components are (2.20) and (2.21) is an equivalence of categories, so the result follows.  $\square$

### 3. CLASSIFICATION OF TORIC DM-STACKS

In this Section we show that the toric DM-stack  $\mathcal{C}_\Delta$  can be viewed as a gerbe banded by a finite abelian group. Then we study the problem of whether two combinatorial data  $\Delta$  and  $\Delta'$  defines isomorphic banded gerbes. We give an answer in combinatorial terms.

Let  $\Delta$  be a combinatorial data as in the introduction, set

$$\Delta_{\text{rig}} := \{N, \Delta, a_\rho | \rho \in \Delta(1)\}.$$

Consider the line bundles  $\mathfrak{V}_1, \dots, \mathfrak{V}_r$  on  $\mathcal{C}_{\Delta_{\text{rig}}}$  defined as follows: for any  $\Delta_{\text{rig}}$ -collection  $(L_\rho, u_\rho, c_m)/Y$ , set

$$\mathfrak{V}_i(Y) := \otimes_\rho L_\rho^{\otimes b_{i\rho}}, \quad i \in \{1, \dots, R\};$$

for any morphism  $(f, \gamma_\rho) : (L'_\rho, u'_\rho, c'_m)/Y' \rightarrow (L_\rho, u_\rho, c_m)/Y$ , set

$$\mathfrak{V}_i(f, \gamma_\rho) := \otimes_\rho \gamma_\rho^{\otimes b_{i\rho}}.$$

Let us denote by  $\sqrt[r_i]{\mathfrak{V}_i}$  the gerbe of  $r_i$ -th roots of  $\mathfrak{V}_i$ , for  $i \in \{1, \dots, R\}$ . Just to fix notation we recall its definition here and refer to [8] Ch. IV (2.5.8.1) and [2] for the general definition and for more details. For a scheme over  $\mathcal{C}_{\Delta_{\text{rig}}}$ ,  $Y$ , an object of  $\sqrt[r_i]{\mathfrak{V}_i}(Y)$  is a pair  $(M, d)$ , where  $M$  is a line bundle on  $Y$  and  $d : M^{r_i} \otimes \mathfrak{V}_i(Y) \rightarrow \underline{\mathbb{C}}$  is an isomorphism. If  $(M', d')$  is an object over  $Y' \rightarrow \mathcal{C}_{\Delta_{\text{rig}}}$ , then a morphism  $(M', d') \rightarrow (M, d)$  over  $(f, \gamma_\rho)$  is given by a morphism of line bundles  $\delta : M' \rightarrow M$  such that the following diagram is cartesian

$$\begin{array}{ccc} M' & \xrightarrow{\delta} & M \\ \downarrow & & \downarrow \\ Y' & \xrightarrow{f} & Y \end{array}$$

and  $d \circ \delta^{\otimes r_i} \otimes_\rho \gamma_\rho^{\otimes b_{i\rho}} = d'$ .

Consider now the functor

$$(3.1) \quad \mathcal{R} : \mathcal{C}_\Delta \rightarrow \mathcal{C}_{\Delta_{\text{red}}}$$

which associates to the  $\Delta$ -collection  $(L_\rho, u_\rho, c_m, M_i, d_i)/Y$  the  $\Delta_{\text{red}}$ -collection  $(L_\rho, u_\rho, c_m)/Y$ , and to  $(f, \gamma_\rho, \delta_i)$  the arrow  $(f, \gamma_\rho)$ . Then we have the following.



**Proposition 3.2.**  $\mathcal{R}$  in (3.1) is a gerbe banded by  $\times_{i=1}^R \mu_{r_i}$  isomorphic to

$$(3.3) \quad \sqrt[r_1]{\mathfrak{V}_1} \times_{\mathcal{C}_{\Delta_{\text{red}}}} \dots \times_{\mathcal{C}_{\Delta_{\text{red}}}} \sqrt[r_R]{\mathfrak{V}_R}.$$

**Proof.** It is straightforward to verify that  $\mathcal{R}$  is a gerbe. Let now  $(L_\rho, u_\rho, c_m, M_i, d_i)/Y$  be a  $\Delta$ -collection. The set of its automorphisms over the identity of  $(L_\rho, u_\rho, c_m)/Y$  is

$$\{(\delta_1, \dots, \delta_R) \in H^0(Y, \mathcal{O}_Y^*)^R \mid \delta_i^{r_i} = 1 \text{ for any } i \in \{1, \dots, R\}\},$$

then the canonical inclusions  $\mu_{r_i} \subset \mathbb{C}^*$ , for  $i \in \{1, \dots, R\}$ , give the  $\mu_{r_1} \times \dots \times \mu_{r_R}$ -banding.

Isomorphism classes of gerbes bounded by  $\mu_{r_1} \times \dots \times \mu_{r_R}$  are in 1-to-1 correspondence with the second cohomology group  $H^2(\mathcal{C}_{\Delta_{\text{red}}}, \times_{i=1}^R \mu_{r_i})$  (see [8]). There is an isomorphism

$$(3.4) \quad H^2(\mathcal{C}_{\Delta_{\text{red}}}, \times_{i=1}^R \mu_{r_i}) \rightarrow \times_{i=1}^R H^2(\mathcal{C}_{\Delta_{\text{red}}}, \mu_{r_i})$$

whose inverse associates the classes of the gerbes  $\mathcal{G}_1, \dots, \mathcal{G}_R$  to the class of the fibered product  $\mathcal{G}_1 \times_{\mathcal{C}_{\Delta_{\text{red}}}} \dots \times_{\mathcal{C}_{\Delta_{\text{red}}}} \mathcal{G}_R$ . The explicit description of (3.4) (as given for example in [8] IV Prop. 2.3.18) shows that the image of the class of  $\mathcal{R}$  is the class of (3.3). This complete the proof.  $\square$

**Remark 3.5.** We can also see  $\mathcal{R}$  as the rigidification of  $\mathcal{C}_\Delta$  with respect to the constant sheaf  $\times_{i=1}^R \mu_{r_i}$  [1] (this accounts for the subscript  $\text{rig}$  in the label  $\Delta_{\text{rig}}$ ).

**Remark 3.6.** The result in Prop. 3.2 could also be obtained directly using the definition of toric DM-stacks given in [4], however in that way we would not obtain the explicit characterization of the gerbe as roots of specified line bundles. More in details the idea is the following. Let  $\Sigma = (N, \Delta, \beta)$  be a stacky fan (it plays the role of our  $\Delta$  but here  $N$  is a finitely generated abelian group). Let us denote by  $N_{\text{free}} := N/N_{\text{tor}}$ , where  $N_{\text{tor}}$  is the torsion part of  $N$ , and by  $\beta_{\text{rig}}$  the composition of  $\beta$  with the projection  $N \rightarrow N_{\text{free}}$ . Set  $\Sigma_{\text{rig}} := (N_{\text{free}}, \Delta, \beta_{\text{rig}})$ . Let us denote by  $\mathcal{X}_\Sigma$  and  $\mathcal{X}_{\Sigma_{\text{rig}}}$  the associated toric DM-stacks. Lemma 2.3 in [4] implies that there is a morphism of stacks

$$\mathcal{X}_\Sigma \rightarrow \mathcal{X}_{\Sigma_{\text{rig}}}.$$

This morphism is a gerbe, and the inclusion  $\text{Hom}_{\mathbb{Z}}(\text{Ext}_{\mathbb{Z}}^1(N_{\text{tor}}, \mathbb{Z}), \mathbb{C}^*) \rightarrow G$  induced by (2.5) in [4] gives a  $\text{Hom}_{\mathbb{Z}}(\text{Ext}_{\mathbb{Z}}^1(N_{\text{tor}}, \mathbb{Z}), \mathbb{C}^*)$ -banding. The choice of an isomorphism  $N_{\text{tor}} \cong \oplus_i \mathbb{Z}/r_i$  induces a  $\times_i \mu_{r_i}$ -banding. Hence we get an isomorphism between  $\mathcal{X}_\Sigma$  and a fibered product of gerbes banded by  $\mu_{r_i}$ , for  $i \in \{1, \dots, R\}$ .

**Theorem 3.7.** For any toric DM-stack  $\mathcal{X}$ , there exists a combinatorial data  $\Delta = \{N, \Delta, a_\rho, r_i, b_{i\rho}\}$  which satisfies the condition

$$(3.8) \quad r_1 | r_2 | \dots | r_R$$

and such that  $\mathcal{X} \cong \mathcal{C}_\Delta$  (here  $r_i | r_{i+1}$  denotes that  $r_i$  divides  $r_{i+1}$ ). Furthermore, if  $\Delta' = \{N, \Delta, a_\rho, r'_i, b'_{i\rho}\}$  is another combinatorial data which satisfies (3.8), then  $\mathcal{C}_\Delta \cong \mathcal{C}_{\Delta'}$  as banded gerbes if and only if

$$R = R', \quad r_i = r'_i \quad \text{for all } i,$$

and the class

$$[\sum_{\rho} (b_{i\rho} - b'_{i\rho}) e_{\rho}^*] \in (\mathbb{Z}^{\Delta(1)})^* / M$$

is divisible by  $r_i$  for any  $i \in \{1, \dots, R\}$ . Here  $M$  is embedded in  $(\mathbb{Z}^{\Delta(1)})^*$  by the dual of the morphism  $\mathbb{Z}^{\Delta(1)} \rightarrow N$  given by  $e_{\rho} \mapsto a_{\rho}$ .

**Proof.** Given  $\mathcal{X}$ , the existence of  $\Delta$  satisfying (3.8) follows from Prop. 3.2 and the classification of finite abelian groups. The condition (3.8) determines the isomorphism class of the generic automorphism group of  $\mathcal{C}_{\Delta}$ . Hence  $\mathcal{C}_{\Delta} \cong \mathcal{C}_{\Delta'}$  implies that  $R = R'$  and  $r_i = r'_i$  for all  $i$ . The same argument in the proof of Prop. 3.2 shows that  $\mathcal{C}_{\Delta} \cong \mathcal{C}_{\Delta'}$  as  $\times_i \mu_{r_i}$ -gerbes if and only if  $\sqrt[r_i]{\mathfrak{V}_i} \cong \sqrt[r_i]{\mathfrak{V}'_i}$  as  $\mu_{r_i}$ -gerbe for any  $i \in \{1, \dots, R\}$ . This is equivalent to the fact that  $\mathfrak{V}_i \otimes \mathfrak{V}'_i{}^{-1}$  is a  $r_i$ -th power of some element in  $\text{Pic}(\mathcal{C}_{\Delta_{\text{red}}})$ . Now the result follows from the fact that  $\text{Pic}(\mathcal{C}_{\Delta_{\text{red}}})$  has the following presentation (see e.g. [5], [7]):

$$0 \rightarrow M \rightarrow (\mathbb{Z}^{\Delta(1)})^* \rightarrow \text{Pic}(\mathcal{C}_{\Delta_{\text{red}}}) \rightarrow 0,$$

where the morphism  $M \rightarrow (\mathbb{Z}^{\Delta(1)})^*$  is the dual of  $\mathbb{Z}^{\Delta(1)} \rightarrow N$ ,  $e_{\rho} \mapsto a_{\rho}$ .  $\square$

#### 4. THE TORUS ACTION

Any toric variety  $X$  contains an algebraic torus  $T$  as open dense subvariety such that the action of  $T$  on itself by multiplication extends to an action on  $X$ . In this Section we show that an analogous property holds for a toric DM-stack once replacing  $T$  with a *Picard stack*  $\mathcal{T}$ . Picard stacks (originally called *champs de Picard*) were defined in Exposé XVIII [11]; we refer to this paper for the definition and further properties. We will denote by  $T$  the torus  $\text{Spec}(\mathbb{C}[M])$ , then any morphism  $g : Y \rightarrow T$  is identified with the corresponding group homomorphism  $M \rightarrow H^0(Y, \mathcal{O}^*)$ ,  $m \mapsto g_m := g^*(\chi^m)$ .

Let us consider

$$\mathcal{T} := \sqrt[r_1]{\mathbb{C}} \times_T \dots \times_T \sqrt[r_R]{\mathbb{C}},$$

and introduce the map

$$(4.1) \quad \mathfrak{m} : \mathcal{T} \times_{(\text{Sch})} \mathcal{T} \rightarrow \mathcal{T}$$

defined on objects by

$$\mathfrak{m}((g, M_i, d_i)/Y, (g', M'_i, d'_i)/Y) := (g \cdot g', M_i \otimes M'_i, d_i \otimes d'_i)/Y,$$

and on arrows by  $\mathfrak{m}(\delta_i, \delta'_i) := \delta_i \otimes \delta'_i$ . The *associativity* is expressed in terms of a natural transformation

$$\sigma : \mathfrak{m} \circ (\mathfrak{m} \times \text{id}_{\mathcal{T}}) \Rightarrow \mathfrak{m} \circ (\text{id}_{\mathcal{T}} \times \mathfrak{m}),$$

and the *commutativity* with a natural transformation

$$\tau : \mathfrak{m} \Rightarrow \mathfrak{m} \circ C,$$

where  $C$  is the functor that exchanges the factors. Here we choose the standard natural transformations  $\sigma$  and  $\tau$ . Then  $(\mathcal{T}, \mathfrak{m}, \sigma, \tau)$  is a Picard stack. Let us denote with  $e$  the neutral element of  $\mathcal{T}$ , which is unique up to unique isomorphism.

The Picard stack  $\mathcal{T}$  acts on  $\mathcal{C}_{\Delta}$ . The action is given by the functor

$$(4.2) \quad \begin{aligned} \mathfrak{a} : \mathcal{C}_{\Delta} \times \mathcal{T} &\rightarrow \mathcal{C}_{\Delta} \\ ((L_{\rho}, u_{\rho}, c_m, M_i, d_i)/Y, (g_m, N_i, e_i)) &\mapsto (L_{\rho}, u_{\rho}, c_m \cdot g_m, M_i \otimes N_i, d_i \otimes e_i)/Y; \end{aligned}$$

and natural transformations  $\alpha : \mathbf{a} \circ (\mathrm{id}_{\mathcal{C}_\Delta} \times \mathbf{m}) \Rightarrow \mathbf{a} \circ (\mathbf{a} \times \mathrm{id}_{\mathcal{T}})$  and  $\beta : \mathrm{id}_{\mathcal{C}_\Delta} \Rightarrow \mathbf{a} \circ (\mathrm{id}_{\mathcal{C}_\Delta} \times e)$  such that the following diagrams are 2-commutative (we put in each square (resp. triangle) the appropriate natural transformation):

$$\begin{array}{ccccc}
 \mathcal{C} \times \mathcal{T} \times \mathcal{T} \times \mathcal{T} & \xrightarrow{\mathrm{id}_{\mathcal{C}} \times \mathrm{id}_{\mathcal{T}} \times \mathbf{m}} & \mathcal{C} \times \mathcal{T} \times \mathcal{T} & & \\
 \downarrow \scriptstyle \mathbf{a} \times \mathrm{id}_{\mathcal{T}} \times \mathrm{id}_{\mathcal{T}} & \searrow \scriptstyle \mathrm{id}_{\mathcal{C}} \times \mathbf{m} \times \mathrm{id}_{\mathcal{T}} & \downarrow \scriptstyle \mathbf{a} \times \mathrm{id}_{\mathcal{T}} & \searrow \scriptstyle \mathrm{id}_{\mathcal{C}} \times \mathbf{m} & \downarrow \scriptstyle \mathrm{id}_{\mathcal{C}} \times \mathbf{m} \\
 & \mathcal{C} \times \mathcal{T} \times \mathcal{T} & \xrightarrow{\mathrm{id}_{\mathcal{C}} \times \mathbf{m}} & \mathcal{C} \times \mathcal{T} & \downarrow \scriptstyle \mathbf{a} \\
 \downarrow \scriptstyle \mathbf{a} \times \mathrm{id}_{\mathcal{T}} & \downarrow \scriptstyle \mathbf{a} \times \mathrm{id}_{\mathcal{T}} & \downarrow \scriptstyle \mathrm{id}_{\mathcal{C}} \times \mathbf{m} & \downarrow \scriptstyle \mathbf{a} & \downarrow \scriptstyle \mathbf{a} \\
 \mathcal{C} \times \mathcal{T} \times \mathcal{T} & \xrightarrow{\mathrm{id}_{\mathcal{C}} \times \mathbf{m}} & \mathcal{C} \times \mathcal{T} & \xrightarrow{\mathbf{a}} & \mathcal{C} \\
 & \downarrow \scriptstyle \mathbf{a} \times \mathrm{id}_{\mathcal{T}} & \downarrow \scriptstyle \mathbf{a} & \downarrow \scriptstyle \mathbf{a} & \\
 & \mathcal{C} \times \mathcal{T} & \xrightarrow{\mathbf{a}} & \mathcal{C} & \\
 & & \downarrow \scriptstyle \mathbf{a} & & \\
 & & \mathcal{C} & & 
 \end{array}$$
  

$$\begin{array}{ccccc}
 \mathcal{C} & & \mathcal{C} & \xrightarrow{\mathrm{id}_{\mathcal{C}} \times e} & \mathcal{C} \\
 \downarrow \scriptstyle \mathrm{id}_{\mathcal{C}} \times e & \searrow \scriptstyle \mathrm{id}_{\mathcal{C}} \times e \times e & \downarrow \scriptstyle \mathrm{id}_{\mathcal{C}} \times e & \searrow \scriptstyle \mathrm{id}_{\mathcal{C}} \times \mathbf{m} & \downarrow \scriptstyle \mathbf{a} \\
 & \mathcal{C} \times \mathcal{T} \times \mathcal{T} & \xrightarrow{\mathrm{id}_{\mathcal{C}} \times \mathbf{m}} & \mathcal{C} \times \mathcal{T} & \\
 & \downarrow \scriptstyle \mathbf{a} \times \mathrm{id}_{\mathcal{T}} & \downarrow \scriptstyle \mathbf{a} & \downarrow \scriptstyle \mathbf{a} & \\
 & \mathcal{C} \times \mathcal{T} & \xrightarrow{\mathbf{a}} & \mathcal{C} & 
 \end{array}$$

There is a standard choice for  $\alpha$  and  $\beta$  which satisfy these conditions.

**Remark 4.3.** The action of a group on a stack has been defined in [9], we refer to this paper for more details. The extension to the case of a group stack is straightforward, the only difference here is that the action  $\mathbf{a}$  must be compatible with the associativity of  $\mathcal{T}$  which is expressed in terms of  $\sigma$ .

**Proposition 4.4.** *There is a morphism*

$$(4.5) \quad \mathcal{T} \rightarrow \mathcal{C}_\Delta$$

whose image is open and dense with respect to the small étale site. The restriction of  $\mathbf{a}$  to  $\mathcal{T}$  with respect to (4.5) is isomorphic to  $\mathbf{m}$ .

**Proof.** Let  $Y \rightarrow \mathcal{T}$  be a morphism given by  $g : Y \rightarrow T$ ,  $M_i$  and  $d_i : M_i^{\otimes r_i} \rightarrow \underline{\mathbb{C}}$ , for  $i \in \{1, \dots, R\}$ . Set  $L_\rho := \underline{\mathbb{C}}$  and  $u_\rho := 1$  for all  $\rho$ ,  $c_m := g_m$ ,  $m \in M$ . Then  $(L_\rho, u_\rho, c_m, M_i, d_i)$  is a  $\Delta$ -collection. This defines the morphism (4.5) on objects, on arrows it sends  $(f, \delta_i)$  to  $(f, \mathrm{id}, \delta_i)$ .

The morphism (4.5) is open with respect to the small étale site. Indeed, let  $\pi : U \rightarrow \mathcal{C}_\Delta$  be an étale covering. It corresponds to a  $\Delta$ -family  $(L_\rho, u_\rho, c_m, M_i, d_i)/U$ . Set  $U' := \{x \in U \mid u_\rho(x) \neq 0, \forall \rho\}$ . The restriction of  $(L_\rho, u_\rho, c_m, M_i, d_i)/U$  to  $U'$  gives a morphism  $U' \rightarrow \mathcal{T}$  such that the following diagram is 2-Cartesian

$$\begin{array}{ccc}
 U' & \longrightarrow & U \\
 \downarrow & & \downarrow \\
 \mathcal{T} & \longrightarrow & \mathcal{C}_\Delta.
 \end{array}$$

The compatibility between  $\mathbf{a}$  and  $\mathbf{m}$  follows directly from the definition. This completes the proof.  $\square$

We proceed by observing that using the definition of [4] it is possible to give another description of the torus action. Indeed let  $\phi : G \rightarrow (\mathbb{C}^*)^n$  be the composition of  $\varphi$  in (2.14) with the projection to  $(\mathbb{C}^*)^n$ . Recall that  $\mathcal{X}_\Sigma = [Z/G]$ , where the  $G$ -action is induced by  $\phi$ . Let us consider the Picard stack  $\mathcal{G} := [(\mathbb{C}^*)^n/G]$  (see 1.4.11 of Exposé XVIII in [11]). Let  $\mathcal{G}^{\text{pre}}$  and  $\mathcal{X}_\Sigma^{\text{pre}}$  be the pre-stacks associated to the groupoids  $((\mathbb{C}^*)^n \times G \rightrightarrows (\mathbb{C}^*)^n)$  and  $(Z \times G \rightrightarrows Z)$  respectively.  $\mathcal{G}^{\text{pre}}$  acts on  $\mathcal{X}_\Sigma^{\text{pre}}$  in the obvious way and the stackification of this action gives an action of  $\mathcal{G}$  on  $\mathcal{X}_\Sigma$ . We denote by  $\mathfrak{a}_\mathcal{X}$  this action.

We want to compare the torus actions previously defined on  $\mathcal{X}_\Sigma$  and on  $\mathcal{C}_\Delta$ . Note that the restriction of (2.16) to  $\mathcal{G}$  gives an isomorphism

$$(4.6) \quad F|_{\mathcal{G}} : \mathcal{G} \rightarrow \mathcal{T}.$$

Moreover there is a natural transformation

$$(4.7) \quad \nu : \mathfrak{a} \circ (F \times F|_{\mathcal{G}}) \Rightarrow F \circ \mathfrak{a}_\mathcal{X}$$

defined as follows. Consider the diagram

$$(4.8) \quad \begin{array}{ccc} \mathcal{X}^{\text{pre}} \times \mathcal{G}^{\text{pre}} & \longrightarrow & \mathcal{C} \times \mathcal{T} \\ \mathfrak{a}_{\mathcal{X}^{\text{pre}}} \downarrow & & \downarrow \mathfrak{a} \\ \mathcal{X}^{\text{pre}} & \longrightarrow & \mathcal{C} \end{array}$$

where each row is the composition of  $(F \times F|_{\mathcal{G}})$  ( $F$  resp.) with the corresponding stackification morphism. There is a canonical natural transformation  $\nu^{\text{pre}}$  which makes (4.8) 2-commutative. Then we define  $\nu$  in (4.7) to be the unique natural transformation induced by  $\nu^{\text{pre}}$ . Finally we have the following.

**Proposition 4.9.** *The isomorphism (2.16) together with  $\nu$  is  $F|_{\mathcal{G}}$ -equivariant.*

**Proof.** Following [9], we have to prove that the diagrams below are 2-commutative with respect to the natural transformations previously defined:

$$\begin{array}{ccccc} \mathcal{X} \times \mathcal{G} \times \mathcal{G} & \xrightarrow{\text{id}_\mathcal{X} \times \mathfrak{m}} & \mathcal{X} \times \mathcal{G} & & \\ \downarrow \mathfrak{a} \times \text{id}_\mathcal{G} & \searrow F \times F| \times F| & \downarrow \mathfrak{a} & \searrow F \times F| & \\ & \mathcal{C} \times \mathcal{T} \times \mathcal{T} & \xrightarrow{\text{id}_\mathcal{C} \times \mathfrak{m}} & \mathcal{C} \times \mathcal{T} & \\ & \downarrow \mathfrak{a} \times \text{id}_\mathcal{T} & \downarrow \mathfrak{a} & \downarrow \mathfrak{a} & \\ \mathcal{X} \times \mathcal{G} & \xrightarrow{\quad \mathfrak{a} \quad} & \mathcal{X} & & \\ & \searrow F \times F| & \downarrow F & \searrow & \\ & \mathcal{C} \times \mathcal{T} & \xrightarrow{\quad \mathfrak{a} \quad} & \mathcal{C} & \end{array}$$

$$\begin{array}{ccccc}
\mathcal{X} \times \mathcal{G} & \xrightarrow{\mathbf{a}} & \mathcal{X} & & \\
& \searrow F \times F| & \nearrow F & & \\
& & \mathcal{C} \times \mathcal{T} & \xrightarrow{\mathbf{a}} & \mathcal{C} \\
& \nearrow \text{id}_{\mathcal{X}} & \nwarrow \text{id}_{\mathcal{C}} & & \\
\mathcal{X} & \xrightarrow{\text{id}_{\mathcal{X}} \times e} & \mathcal{C} \times \mathcal{T} & \xrightarrow{\text{id}_{\mathcal{C}}} & \mathcal{C} \\
& \searrow F & \nearrow \text{id}_{\mathcal{C}} & & \\
& & \mathcal{C} & & 
\end{array}$$

With abuse of notation, we have denoted with the same  $\mathbf{m}$  the two multiplications of the Picard stacks and with the same  $\mathbf{a}$  the two actions. Notice that it is enough to prove the 2-commutativity of the same diagrams but with  $\mathcal{X}^{\text{pre}}$  and  $\mathcal{G}^{\text{pre}}$  instead of  $\mathcal{X}$  and  $\mathcal{G}$  respectively. Now this can be verified by direct computations.  $\square$

## 5. MORPHISMS BETWEEN TORIC STACKS

The content in Thm. 2.5 can be refined when  $\mathcal{Y}$  is a toric stack too. In this Section we give a description of morphisms between toric DM-stacks parallel to the one given in Thm. 3.2 of [6] in the context of toric varieties. We need some introductory notations. Let  $\mathcal{Y} := \mathcal{C}_{\Delta'}$  be the toric DM-stack defined by  $\Delta' := \{N', \Delta', a'_\rho\}$  (notice that here we set  $R' = 0$ ). We assume that the coarse moduli space  $Y$  of  $\mathcal{Y}$  is a complete variety. Let  $\mathcal{X} := \mathcal{C}_{\Delta}$  be a toric DM-stack such that the 1-dimensional rays generate  $N_{\mathbb{Q}}$ . Let us fix the presentations  $\mathcal{Y} = [Z'/G']$  and  $\mathcal{X} = [Z/G]$  as in [4].

The Picard group of  $\mathcal{Y}$ ,  $\text{Pic}(\mathcal{Y})$ , is isomorphic to the group of characters of  $G'$  ([5], [7]). The isomorphism associates to the character  $\chi$  the isomorphism class  $[\mathcal{L}(\chi)] \in \text{Pic}(\mathcal{Y})$  of the trivial line bundle on  $Z'$  with the  $G'$ -linearization given by  $\chi$ . We use this isomorphism to identify the two groups.

Considered  $\mathcal{Y}$ , we recall that there are distinguished elements  $[\mathcal{L}_\rho] \in \text{Pic}(\mathcal{Y})$ , for  $\rho \in \Delta'(1)$ . Let  $\phi' : G' \rightarrow (\mathbb{C}^*)^{\Delta'(1)}$  be the component of  $\varphi'$  in (2.14) which maps to  $(\mathbb{C}^*)^{\Delta'(1)}$ , the components of  $\phi'$ ,  $(\phi')_\rho$  for  $\rho \in \Delta'(1)$ , are characters of  $G'$ , then the  $[\mathcal{L}_\rho]$ 's are the corresponding isomorphism classes of line bundles.

Consider the homogeneous coordinate ring of  $\mathcal{Y}$  defined in [7] as the polynomial ring  $S^{\mathcal{Y}} := \mathbb{C}[z_\rho]$  in the variables  $z_\rho$ , where  $\rho \in \Delta'(1)$ . It is endowed with a  $\text{Pic}(\mathcal{Y})$ -grading: the monomial  $\prod_\rho z_\rho^{l_\rho}$  has degree  $\sum_\rho [\mathcal{L}_\rho]^{l_\rho} \in \text{Pic}(\mathcal{Y})$ . For any  $\chi \in \text{Pic}(\mathcal{Y})$ , let us denote by  $S^\chi_{\mathcal{Y}}$  the subset of  $S^{\mathcal{Y}}$  consisting of homogeneous polynomials of degree  $\chi$ . There is an isomorphism of complex vector spaces (see [7])

$$H^0(\mathcal{Y}, \mathcal{L}(\chi)) \cong S^\chi_{\mathcal{Y}}.$$

With this notations we can now state the following result.

**Theorem 5.1.** *Let  $P_\rho \in S^{\mathcal{Y}}$  be homogeneous polynomials indexed by  $\rho \in \Delta(1)$ , and let  $\chi_i \in \text{Pic}(\mathcal{Y})$ , for  $i \in \{1, \dots, R\}$ , such that:*

- (a): *If  $P_\rho \in S^{\mathcal{Y}}_{\chi_\rho}$ , then  $\sum_\rho \chi_\rho \otimes a_\rho = 0$  in  $\text{Pic}(\mathcal{Y}) \otimes_{\mathbb{Z}} N$ , and  $\prod_\rho \chi_\rho^{b_{i\rho}} \cdot \chi_i^{r_i} = 1$  in  $\text{Pic}(\mathcal{Y})$  for any  $i$ ;*
- (b):  *$(P_\rho(z)) \notin Z$  whenever  $z \notin Z'$ .*

If we define  $\tilde{f}(z) = (P_\rho(z)) \in \mathbb{C}^{\Delta(1)}$ , then there is a morphism  $f : \mathcal{Y} \rightarrow \mathcal{X}$  such that the diagram

$$\begin{array}{ccc} \mathbb{C}^{\Delta'(1)} - Z' & \xrightarrow{\tilde{f}} & \mathbb{C}^{\Delta(1)} - Z \\ \downarrow & & \downarrow \\ \mathcal{Y} & \xrightarrow{f} & \mathcal{X} \end{array}$$

is 2-commutative, where the vertical arrows are the quotient maps. Furthermore:

- (i): Let  $\chi_i \in \text{Pic}(\mathcal{Y})$  fixed for  $i \in \{1, \dots, R\}$ . Then two sets of polynomials  $\{P_\rho\}$  and  $\{P'_\rho\}$  determine 2-isomorphic morphisms if and only if there exists  $g \in G$  such that  $P'_\rho = \varphi_\rho(g)P_\rho$  for any  $\rho \in \Delta(1)$ , where  $\varphi_\rho$  is the  $\rho$ -th component of  $\varphi$  defined in (2.14).
- (ii): All morphisms  $f : \mathcal{Y} \rightarrow \mathcal{X}$  arise in this way.

**Proof.** Let us choose a representative  $\mathcal{L}(\chi)$  of the class  $\chi \in \text{Pic}(\mathcal{Y})$  for any  $\chi$ . Note that there are canonical isomorphisms  $\mathcal{L}(\chi_1) \otimes \mathcal{L}(\chi_2) \cong \mathcal{L}(\chi_1 \cdot \chi_2)$ .

Let  $\{P_\rho | \rho \in \Delta(1)\}$  and  $\{\chi_i | i \in \{1, \dots, R\}\}$  satisfying (a) and (b). Then  $\prod_\rho \chi_\rho^{\langle m, a_\rho \rangle} = 1$ , for any  $m \in M$ . As a consequence there are canonical isomorphisms

$$c_m^{\text{can}} : \otimes_\rho \mathcal{L}(\chi_\rho)^{\otimes \langle m, a_\rho \rangle} \rightarrow \underline{\mathbb{C}}, \quad m \in M,$$

and

$$d_i^{\text{can}} : \otimes_\rho \mathcal{L}(\chi_\rho)^{\otimes b_{i\rho}} \otimes \mathcal{L}(\chi_i)^{\otimes r_i} \rightarrow \underline{\mathbb{C}}, \quad i \in \{1, \dots, R\}.$$

It follows that  $(\mathcal{L}(\chi_\rho), P_\rho, c_m^{\text{can}}, \mathcal{L}(\chi_i), d_i^{\text{can}})/\mathcal{Y}$  is a  $\Delta$ -collection, therefore it corresponds to a morphism  $f : \mathcal{Y} \rightarrow \mathcal{X}$ . The commutativity of the diagram follows easily.

Let now  $\{P_\rho\}$  and  $\{P'_\rho\}$  be two sets of polynomials defining 2-isomorphic morphisms. Then there are isomorphisms

$$\gamma_\rho : \mathcal{L}(\chi_\rho) \rightarrow \mathcal{L}(\chi_\rho) \quad \text{and} \quad \delta_i : \mathcal{L}(\chi_i) \rightarrow \mathcal{L}(\chi_i),$$

such that for any  $\rho, m, i$ ,

$$\gamma_\rho(P_\rho) = P'_\rho, \quad \otimes_\rho \gamma_\rho^{\otimes \langle m, a_\rho \rangle} = \text{id}, \quad \otimes_\rho \gamma_\rho^{b_{i\rho}} \otimes \delta_i^{r_i} = \text{id}.$$

The  $\gamma_\rho$ 's and  $\delta_i$ 's are multiplications by non-zero complex numbers which we denote by the same symbols. The previous conditions means that  $((\gamma_\rho)_\rho, (\delta_i)_i) \in \ker(\psi)$ , where  $\psi$  is defined in (2.14). So there exists  $g \in G$  such that  $\varphi(g) = ((\gamma_\rho)_\rho, (\delta_i)_i)$ .

To conclude the proof, let  $(\mathcal{L}_\rho, u_\rho, c_m, M_i, d_i)/\mathcal{Y}$  be a  $\Delta$ -collection. There is a morphism

$$(\mathcal{L}_\rho, u_\rho, c_m, M_i, d_i) \rightarrow (\mathcal{L}(\chi_\rho), P_\rho, \tilde{c}_m, \mathcal{L}(\chi_i), \tilde{d}_i)$$

for some  $\chi_\rho, \chi_i \in \text{Pic}(\mathcal{Y})$ . Clearly the  $P_\rho$ 's,  $\chi_\rho$ 's and  $\chi_i$ 's satisfy conditions (a) and (b). Let us now consider the automorphisms  $(c_m^{\text{can}})^{-1} \circ \tilde{c}_m$  and  $(d_i^{\text{can}})^{-1} \circ \tilde{d}_i$  of  $\otimes_\rho \mathcal{L}(\chi_\rho)^{\otimes \langle m, a_\rho \rangle}$  and  $\otimes_\rho \mathcal{L}(\chi_\rho)^{b_{i\rho}} \otimes \mathcal{L}(\chi_i)^{r_i}$  respectively. They correspond to an element  $\left( ((c_m^{\text{can}})^{-1} \circ \tilde{c}_m)_m, ((d_i^{\text{can}})^{-1} \circ \tilde{d}_i)_i \right) \in (\mathbb{C}^*)^d \times (\mathbb{C}^*)^R$ . Let now  $((\gamma_\rho)_\rho, (\delta_i)_i) \in (\mathbb{C}^*)^n \times (\mathbb{C}^*)^R$  such that  $\psi((\gamma_\rho)_\rho, (\delta_i)_i) = \left( ((c_m^{\text{can}})^{-1} \circ \tilde{c}_m)_m, ((d_i^{\text{can}})^{-1} \circ \tilde{d}_i)_i \right)$ . Then (2.15) implies that

$$(\gamma_\rho, \delta_i) : (\mathcal{L}(\chi_\rho), P_\rho, \tilde{c}_m, \mathcal{L}(\chi_i), \tilde{d}_i) \rightarrow (\mathcal{L}(\chi_\rho), P_\rho, c_m^{\text{can}}, \mathcal{L}(\chi_i), d_i^{\text{can}})$$

is a morphism. By construction, the morphism associated to  $(\mathcal{L}(\chi_\rho), P_\rho, c_m^{\text{can}}, \mathcal{L}(\chi_i), d_i^{\text{can}})$  is 2-isomorphic to  $f$ .  $\square$

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